MATRICES

Chapter 3

THEORY CONTENT OF MATRICES

1 DEFINITION OF A MATRIX

A system of '*mn*' numbers (real or complex) arranged in a rectangular array of *m* rows and *n* columns is called a matrix. This system can be arranged in any of the following patterns.

$$\left(\quad\right);\ \left[\quad\right];\ \left\|\quad\right\|$$

In general a_{ij} represent the element (or entry) of i^{th} row and j^{th} column, so the matrix can be represented as (a_{ij}) or $[a_{ij}]$ or $||a_{ij}||$

2 ORDER OF A MATRIX

If any matrix *A* contains '*m*' rows and '*n*' columns then $m \times n$ is termed as order of matrix.

Order is generally written as suffix of the array.

Now any matrix of order $m \times n$ will have the notation $[a_{ij}]_{m \times n}$.

i.e. $A = [a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$ or $\|a_{ij}\|_{m \times n}$

it is obvious that $1 \le i \le m$ and $1 \le j \le n$

Illustration 1

Question: **In the inter sports meet of local colleges the games to be played are T.T., Hockey, Badminton, Tennis, and B. Ball. The three colleges of Meerut sent the following number of players.**

Meerut College (*M*.C.) – 35 players ; 5(T. T.), 11 (Hockey), 5(Bad), 6 (Tennis) and 8(B. Ball).

Nanak Chand College (N. A. S.) – 22 players ; 3(T. T), 13 (Hockey), 2 (Bad), 4 (Tennis) and none for (B. Ball).

Dev Nagri College (D. N.) – 31 players ; 2(T. T.), 15 (Hockey) 3(Bad), 5 (Tennis) and 6 (B. Ball). Put this information in matrix form.

Solution: The above information can be put in tabular form as under.

Colleges	Number of players				
	T.T.	Hockey	Badminton	Tennis	B. Ball
M. C. (35)	5	11	5	6	8
N. A. S. (22)	3	13	2	4	0
D. *N*. (31)	2	15	3	5	6

The number 4 represents the number of players the N.A.S. College has sent for playing Tennis. The number 15 represents the number of players the D.N. college has sent for playing Hockey. Similarly, the numbers 8 represents the number of players which Meerut College has sent for playing basket ball. The above can be put in rectangular array form as

$$\begin{bmatrix} 5 & 11 & 5 & 6 & 8 \\ 3 & 13 & 2 & 4 & 0 \\ 2 & 15 & 3 & 5 & 6 \end{bmatrix}$$

Above is a 3×5 matrix, 3 represents the number of rows (number of colleges) participating and 5 represents the number of games being played in the meet.

3 TYPES OF MATRIX

The elements which appear in the rectangular array are known as entries ; depending upon these entries, matrices are of following types:

3.1 ROW MATRIX

A single row matrix is called a row matrix or a row vector.

e.g. the matrix $[a_{11} a_{12} \ldots.. \; a_{1n}]$ is a $1 \times n$ row matrix.

3.2 COLUMN MATRIX

A single column matrix is called a column matrix or a column vector.

e.g. the matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ \ldots \\ a_{mj} \end{bmatrix}$ is a $m \times 1$ column matrix.

3.3 SQUARE MATRIX

If $m = n$, i.e. if the number of rows and columns of a matrix are equal say n, then it is called a square matrix of order n.

3.4 NULL (or zero) MATRIX

If all the elements of a matrix are equal to zero, then it is called a null matrix and is denoted by $O_{m \times n}$ or O.

3.5 DIAGONAL MATRIX

A square matrix in which all its elements are zero except those in the leading diagonal, is called a diagonal matrix. Thus in a diagonal matrix $a_{ij} = 0$ if $i \neq j$.

The diagonal matrices of order 2 and 3 are as follows:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

The elements a_{ij} of a matrix for which $i = j$ are called the diagonal elements of a matrix and the diagonal along which all these elements lie is called the principal diagonal or the diagonal of the matrix.

3.6 SCALAR MATRIX

A square matrix in which all the diagonal elements are equal and all other elements equal to zero is called a scalar matrix.

i.e. in a scalar matrix $a_{ij} = k$, for $i = j$ and $a_{ij} = 0$ for $i \neq j$. Thus $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix.

3.7 UNIT MATRIX OR IDENTITY MATRIX

A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or identity matrix.

e.g. a unit (or identity) matrix of order 2 and 3 are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3.8 NEGATIVE OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$ be a matrix. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.

4 EQUALITY OF MATRICES

Two matrices A and B are said to be equal, written as $A = B$, if,

(i) they both are of the same order i.e. have the same number of rows and columns, and

(ii) the elements in the corresponding places of the two matrices are the same.

5 ADDITION AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same type $m\times n$. Then their sum (or difference) $A + B$ (or $A–B$) is defined as another matrix of the same type, say $C = [c_{ij}]$ such that any element of C is the sum (or difference) of the corresponding elements of A and B.

$\therefore \quad C = A\pm B = [a_{ij}\pm b_{ij}]$

Illustration 2

Question: $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$

Solution: Here both A and B are 2×3 matrices

$$\therefore \quad A+B = \begin{bmatrix} 1+7 & 2+3 & 4+2 \\ 0+5 & 5+1 & 3+9 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$$

$$\text{and} \quad A-B = \begin{bmatrix} 1-7 & 2-3 & 4-2 \\ 0-5 & 5-1 & 3-9 \end{bmatrix} = \begin{bmatrix} -6 & -1 & 2 \\ -5 & 4 & -6 \end{bmatrix}$$

5.1 PROPERTIES OF MATRIX ADDITION

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $k(A + B) = kA + kB$ here k is any scalar.
4. $A + O = O + A = A$, here O {null matrix} will be additive identity.
5. If A be a given matrix then the matrix $–A$ is the additive inverse of A for $A + (–A) =$ null matrix O.
6. If A, B and C be three matrices of the same type
   then $A + B = A + C \Rightarrow B = C$ (Left Cancellation Law)
   and $B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

6 MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k a scalar. Then the matrix obtained by multiplying each element of matrix A by k is called the scalar multiple of A and is denoted by kA.

6.1 PROPERTIES

- If k_1 and k_2 are scalars and A be a matrix, then $(k_1 + k_2) A = k_1A + k_2A$.
- If k_1 and k_2 are scalars and A be a matrix, then $k_1(k_2A) = (k_1k_2)A$.
- If A and B are two matrices of the same order and k, a scalar,
then $k(A + B) = kA + kB$.
  i.e. the scalar multiplication of matrices distributes over the addition of matrices.
- If A is any matrix and k be a scalar, then $(–k)A = –(kA) = k(–A)$.

7 MULTIPLICATION OF TWO MATRICES

Let $A = [a_{ij}]$ be $m\times p$ matrix and $B = [b_{ij}]$ be $p\times n$ matrix. These matrices A and B are such that the number of columns of A are the same as the number of rows of B each being equal to p. Then the product AB (in the order it is written) will be a matrix $C = [c_{ij}]$ of the type $m\times n$.

Where c_{ij} will be the element of C occurring in i^{th} row and j^{th} column and it will be row by column product of i^{th} row of A having p columns with j^{th} column of B having p rows, the elements of which are

a_{i1} a_{i2} ….. a_{ip} and b_{1j}
$a_{i1}a_{i2}$ ….. a_{ip} and b_{2j}
………………………
………………………

$\therefore \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots.. a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

The summation is to be performed w.*r*.t. repeated suffix *k*.

Above gives us the particular *i*-*j*th element of C which is $m \times n$ type. For getting an element of C occurring in 2nd row and 3rd column we shall put $i = 2$ and $j = 3$.

$$\therefore \quad c_{23} = \sum_{k=1}^{p} a_{2k}b_{k3} = a_{21}b_{13} + a_{22}b_{23} + \ldots\ldots. a_{2p}b_{p3}$$

There being *m* rows in *A*, *i* can take values from 1 to *m* and there being *n* columns in *B*, *j* can take values from 1 to *n*, and thus we shall get all the *mn* elements of *C*.

Again $\quad c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \quad \ldots$(i)

Above gives us *i*-*j*th element of *AB* which is of $m \times n$ type having *m* rows and *n* columns.

7.1 ELEMENTS OF *j*th COLUMN OF AB

For getting elements of *j*th column, *j* will remain fixed for *j*th column whereas *i* will change from 1 to *m* as there are *m* rows in *AB*.

Hence giving *i* the values 1, 2, 3……*m* and keeping *j* fixed in (i) we shall get all the elements of *j*th column of *AB*.

$\therefore \quad$ *j*th column of *AB* is $\sum_{k=1}^{p} a_{1k}\ b_{kj}\ , \sum_{k=1}^{p} a_{2k}b_{kj}\ , \ldots.. , \sum_{k=1}^{p} a_{mk}\ b_{kj}$

7.2 AN EASY WAY TO REMEMBER

If we denote the ordered set of rows of *A* by R_1, R_2, R_3 each having 2 elements and ordered set of columns of *B* by C_1, C_2, each having 2 elements.

Then $\quad AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}_{3\times1} [C_1 C_2]_{1\times2} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}_{3\times2}$

7.3 FEW IMPORTANT THINGS FOR THE MULTIPLICATION

1. **Condition for product AB to exist or to be defined:** If *A* and *B* be two matrices then their product is defined or in other words *A* is *conformable* to *B* for multiplication if the number of columns of *A* is the same as the number of rows in *B*. i.e. If *A* be $m \times p$ and *B* be p $\times n$, the matrix *AB* will be of the type $m \times n$.
2. **Pre-multiplication and post multiplication**
   When we say multiply *A* by *B* then it could mean both *AB* or *BA* where *A* and *B* are any numbers. But when *A* and *B* are matrices then as seen above *AB* and BA do not necessarily mean the same thing. If *AB* is defined for matrix multiplication *BA* may not be defined. To avoid this when we say product *AB* it would mean the matrix *A* post-multiplied by *B* and when we say product *BA* it would mean matrix *A* pre-multiplied by *B*. In *AB*, *A* is called *prefactor*and *B**post factor*.
3. In the case when both *A* and *B* are square matrices of the same type then also both *AB* and *BA* are defined and the product matrix is also a matrix of the same type but still $AB \neq BA$.
4. Again we know that when $ab = 0$ it means that either *a* or *b* (or both) is zero. But $AB = O$ i.e. a null matrix does not necessarily imply that either *A* or *B* = O as shown above because neither *A* nor *B* is null matrix whereas *AB* is a null matrix.

Illustration 3

Question: **If** $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ **compute** AB **and** BA**.**

Solution: Here A is 3×3 and B is 3×3. Hence both AB and BA are defined and each will be 3×3 matrix.

Let $AB = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

where C_{ij} means that take the product of ith row of A with jth column of B.

e.g. C_{23} = product of 2nd row of A with 3rd column of B.

i.e. $[-3 \quad 2 \quad -1] \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = -3.3 + 2.6 - 1.3 = 0$.

Similarly we can find other elements of C.

We can also say that by the product of first row of A with the three columns of B ; we shall get the three elements of first row of C.

i.e. R_1C_1, R_1C_2, R_1C_3

and similarly take the second row of A and multiply with all the columns of B and you will get the three elements of 2nd row of C i.e. R_2C_1, R_2C_2, R_2C_3 and elements of 3rd row of C will be R_3C_1, R_3C_2, R_3C_3.

$$\therefore \quad AB = \begin{bmatrix} 1.1 - 1.2 + 1.1 & 1.2 - 1.4 + 1.2 & 1.3 - 1.6 + 1.3 \\ -3.1 + 2.2 - 1.1 & -3.2 + 2.4 - 1.2 & -3.3 + 2.6 = 1.3 \\ 2.1 + 1.2 + 0.1 & -2.2 + 1.4 + 0.2 & -2.3 + 1.6 + 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \text{ (i.e. , null matrix)}$$

Similarly BA can also be computed.

Illustration 4

Question: **If A and B be matrices such that both, AB and $A + B$ are defined. Prove that both A and B are square matrices of the same order.**

Solution: We know that two matrices A and B are conformable for addition if they are of the same type. Thus if A be $m \times n$ then B should also be $m \times n$ as $A + B$ is defined.

Again since AB is also defined therefore number of columns in A i.e., n should be equal to number of rows in B i.e. m. Hence $n = m$ and in that case both A and B will be square matrices of order equal to $m = n$.

Illustration 5

Question: **If A be any $m \times n$ matrix and both AB and BA are defined prove that B should be $n \times m$ matrix.**

Solution: Since A is $m \times n$ and AB isdefined, therefore B should be $n \times$ p because the number of columns of A should be equal to number of rows of B.

Again B is now $n \times$ p and A is $m \times n$.

And since BA is also defined therefore p would be equal to m by the same argument as above.

$\therefore$ B is $n \times m$ matrix.

7.4 PROPERTIES OF MATRIX MULTIPLICATION

(a) Multiplication of matrices is distributive with respect to addition of matrices
i.e. $A(B + C) = AB + AC$.

(b) Matrix multiplication is associative if conformability is assured.
i.e. $A(BC) = (AB)C$.

(c) The multiplication of matrices is not always commutative.
i.e. AB is not always equal to BA.

(d) Multiplication of a matrix A by a null matrix conformable with A, will give null-matrix.

i.e. Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 2 \\ 6 & 4 & 2 \\ 7 & 4 & 6 \end{bmatrix}_{4\times3}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}$ $\Rightarrow AO = O$

8 OPERATIONS REGARDING MATRICES

8.1 TRANSPOSE OF A MATRIX

If A be a given matrix of the type $m\times n$ then the matrix obtained by changing the rows of A into columns and columns of A into rows is called transpose of matrix A and is denoted by A' or A^T. As there are m rows in A therefore there will be m columns in A' and similarly as there are n columns in A there will be n rows in A'.

Properties of transpose

(i) $(A')' = A$

(ii) $(KA)' = KA'$. K being a scalar.

(iii) $(A \pm B)' = A' \pm B'$

(iv) $(AB)' = B'A'$.

(v) $(ABC)' = C'\,B'A'$.

8.2 THE CONJUGATE OF A MATRIX

Let $A = [a_{ij}]$ be a given matrix then the matrix obtained by replacing all the elements by their conjugate complex is called the conjugate of matrix A and will be represented by $\overline{A}$ i.e. $\overline{A} = [\overline{a}_{ij}]$.

(i) $\overline{\overline{A}} = A$

(ii) $\left(\overline{A+B}\right) = \overline{A} + \overline{B}$

(iii) $(\overline{AB}) = \overline{A}\,\overline{B}$

8.3 TRANSPOSE OF THE CONJUGATE OF A MATRIX

Transpose of the conjugate of a matrix is equal to the conjugate of the transpose of a matrix A i.e. $(\overline{A})' = (\overline{A'})$ and written as A^θ.

(i) $(A^\theta)^\theta = A$

(ii) $(A + B)^\theta = A^\theta + B^\theta$

(iii) $(KA)^\theta = \overline{K}A^\theta$, K being a scalar

(iv) $(AB)^\theta = B^\theta A^\theta$

8.4 MINOR OF ANY ELEMENT OF A MATRIX

Consider the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If we leave the row and the column passing through the element a_{ij}, then the second order determinant thus obtained is called the minor of the element a_{ij} and we shall denote it by M_{ij}. In this way we can get 9 minors corresponding to the 9 elements of Δ.

For example:

the minor of the element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$,

the minor of the element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$,

the minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$, and so on.

8.5 COFACTOR OF ANY ELEMENT OF A MATRIX

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called cofactor of the element a_{ij}. We shall denote the cofactor of an element by the C_{ij}. With this notation, cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$.

8.6 DETERMINANT OF ANY MATRIX

If matrix $A = [a_{ij}]$ is a square matrix of order 'n' then

$$\text{determinant of } A = \left(\sum_{k=1}^{n} a_{1k}C_{1k}\right) = \left(\sum_{k=1}^{n} a_{2k}C_{2k}\right) = \ldots\ldots\ldots = \ldots\ldots\ldots$$

$$= \left(\sum_{k=1}^{n} a_{k1}C_{k1}\right) = \left(\sum_{k=1}^{n} a_{k2}C_{k2}\right) = \ldots\ldots\ldots = \ldots\ldots\ldots$$

here C_{ik} represents cofactor of the element of i^{th} row and k^{th} column of matrix A.

for 3×3 order matrix A ; det A (or $|A|$) $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

8.7 TRACE OF A MATRIX

Let A be a square matrix of order n. The sum of the elements of A lying along the principal diagonal is called the trace of A. We shall write the trace of A as $tr\ A$. Thus if $A = [a_{ij}]_{n\times n}$, then $tr\ A = \sum_{i=1}^{n} a_i = a_{11} + a_{22} + \ldots\ldots + a_{nn}$.

9 TYPES OF MATRIX ON THE BASIS OF OPERATIONS

9.1 SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ element is the same as its (j, i)th element i.e., if $a_{ij} = a_{ji}$ for all i, j.

9.2 SKEW SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if the $(i, j)^{th}$ element of A is the negative of the (j, i)th element of A i.e., if $a_{ij} = -a_{ji}$ for all i, j.

9.3 HERMITIAN MATRIX

A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i, j)^{th}$ element of A is equal to conjugate complex of the (j, i)th element of A i.e., if $a_{ij} = \bar{a}_{ji}$ for all i and j.

9.4 SKEW HERMITIAN MATRIX

A square matrix $A = [a_{ij}]$ is said to be Skew Hermitian if the $(i, j)^{th}$ element of A is equal to the negative of conjugate complex of the (j, i)th element of A i.e., if $a_{ij} = -\bar{a}_{ji}$ for all i and j.

9.5 ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $A'A = I = AA'$

9.6 UNITARY MATRIX

A square matrix A is said to be unitary if $A^{\theta}A = I = AA^{\theta}$

9.7 IDEMPOTENT MATRIX

A matrix such that $A^2 = A$ is called idempotent matrix.

9.8 NILPOTENT MATRIX

A matrix A will be called a nilpotent matrix if $A^k = O$ (null matrix) where k is a positive integer. If however k is the least positive integer for which $A^k = O$ then k is the *index* of the nilpotent matrix A.

9.9 INVOLUTRY MATRIX

A matrix A such that $A^2 = I$ is called involutry matrix.

PROFICIENCY TEST–I

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$ then $AB + BA = O$. (True/False)

2. If $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$ then $x = -3$, $y = -2$, and $z = 4$. (True/False)

3. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$, then $A^3 - 3A^2 - A + 9I =$ …….

4. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $A.A' = I$, then find $x + y$.

5. For the 3 matrices A, B, C, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ verify the following relations:

$A^2 = B^2 = C^2 = I$

$AB = -BA$; $AC = -CA$; $BC = -CB$.

ANSWERS TO PROFICIENCY TEST–I

1. False

2. True

3. zero

4. –3

10 ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n\times n}$ be any $n\times n$ matrix. The transpose B' of the matrix $B = [C_{ij}]_{n\times n}$, where C_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$, is called the adjoint of the matrix A and is denoted by the symbol *adjA*.

Illustration 6

Question: **If** $A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$ **, then find *adj* A.**

Solution: In $|A|$, the cofactor of α is δ and the cofactor of β is $-\gamma$. Also the cofactor of γ is $-\beta$ and the cofactor of δ is α. Therefore the matrix B formed of the cofactor of the elements of $|A|$ is

$$B=\begin{bmatrix}\delta & -\gamma\\ -\beta & \alpha\end{bmatrix}$$

Now Adj A = the transpose of the matrix $B=\begin{bmatrix}\delta & -\beta\\ -\gamma & \alpha\end{bmatrix}$

11 INVERSE OF A MATRIX

Let A be any n-rowed square matrix. Then a matrix B, if it exists, such that $AB = BA = I_n$ is called inverse of A.

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

If A be an invertible matrix, then the inverse of A is $\dfrac{1}{|A|}$ Adj. A. It is usual to denote the inverse of A by A^{-1}.

- **Theorem (Uniqueness of inverse)**

Inverse of a square matrix if it exists is unique.

Proof:

Let $A=[a_{ij}]_{n\times n}$ be a square matrix. Let inverse of A exist.

To prove inverse of A is unique:

If possible, let B and C be two inverses of A

Then $\quad AB = BA = I_n$

And $\quad AC = CA = I_n$

Now $B = BI_n$

$= B(AC) \quad [\because\ AC = I_n]$

$= (BA)C = I_nC = C$

Hence $B = C \Rightarrow$ inverse of A is unique

11.1 PROPERTIES

(i) $(AB)^{-1} = B^{-1}A^{-1}$,

(ii) $(A')^{-1} = (A^{-1})'$

(iii) $(A^{-1})^{\theta} = (A^{\theta})^{-1}$

Illustration 7

Question: **Find the inverse of the matrix** $A=\begin{bmatrix}\mathbf{0} & \mathbf{1} & \mathbf{2}\\ \mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{3} & \mathbf{1} & \mathbf{1}\end{bmatrix}$.

Solution: We have

$$|A|=\begin{vmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{vmatrix}=\begin{vmatrix}0 & 1 & 0\\ 1 & 2 & -1\\ 3 & 1 & -1\end{vmatrix}\text{, applying } C_3\to C_3-2C_2$$

$= -1\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$, expanding the determinant along the first

$= -2$

Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

i.e. , are –1 , 8 , –5 respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

i.e. , are 1, –6, 3 respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

i.e. , are –1, 2, –1 respectively.

Therefore the Adj. A = the transpose of the matrix B where

$$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \therefore \text{ Adj. } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now $A^{-1} = \dfrac{1}{|A|}$ Adj. A and here $|A| = -2$.

$$\therefore \quad A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

12 SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be non-singular or singular according as $|A| \neq 0$ or $|A| = 0$.

13 ELEMENTARY OPERATIONS OR ELEMENTARY TRANSFORMATIONS OF A MATRIX

Definition:

Any of the following operations is called an **elementary transformation (operation).**

(i) The interchange of any two rows (or columns).

(ii) The multiplication of the elements of any row (or column) by a non-zero number.

(iii) The addition to the elements of any row (or column), the corresponding elements of any other row (or column) multiplied by a non-zero number.

Any elementary transformation is called a row transformation or column transformation according as it applies to rows or columns.

Clearly, there will be a total of six elementary operations (transformations) on a matrix, three of them are due to rows and are called row operations where as three of them are due to columns and are called column operations.

1. The elementary operations of interchange of *i*th row and *j*th row is denoted by $R_i \leftrightarrow R_j$ and interchange of *i*th column and *j*th column is denoted by $C_i \leftrightarrow C_j$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$ i.e., interchanging Ist row and 3rd row matrix A becomes the matrix

$$B = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

2. The elementary operation of multiplication of the elements of the *i*th row by a non-zero number *k* is denoted by $R_i \to kR_i$.

Similarly, the multiplication of the elements of the *i*th column by a non-zero number *k* is denoted by $C_i \to kC_i$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

On multiplying the elements of 3rd column of matrix *A* by 2, i.e., on applying $C_3 \to 2C_3$, we get the new matrix

$$B = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 8 \\ 2 & 0 & 10 \end{bmatrix}$$

3. The elementary operation of the addition to the elements of the *i*th row, the corresponding elements of the *j*th row multiplied by a non-zero number *k* is denoted by $R_i \to R_i + kR_j$.

Similarly, the elementary operation of the addition to the elements of the *i*th column, the corresponding elements of the *j*th column multiplied by a non-zero number *k* is denoted by $C_i \to C_i + kC_j$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 4 \\ 2 & 4 & 5 \end{bmatrix}$

On applying the elementary operation $C_2 \to C_2 + 2C_1$, matrix *A* becomes the matrix *B*.

- **Equivalent Matrices**

Two matrices *A* and *B* are said to be **equivalent** if one can be obtained from other by applying a finite number of elementary operations on the other matrix. If *A* and *B* are equivalent matrices we write $A \sim B$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix}$

Now $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}$ [Applying $R_1 \leftrightarrow R_2$]

$\sim \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix} = B$ [Applying $C_3 \to 2C_3$]

Here $A \sim B$ as *B* has been obtained from *A* by applying two elementary operations.

- **Elementary Matrix:**

A matrix obtained from unit matrix by a single elementary operation is called an elementary matrix.

Example:

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $[R_1 \to 2R_1]$

is an elementary matrix.

Inverse of a matrix by elementary operations (Elementary operations on matrix equation)

Let A, B and X be three matrices of the same order such that

$X = AB$ ...(i)

The matrix equation (i) will be also valid if we apply a row operation on matrix X [occurring on the L.H.S. of equation (i)] and the same row operation on matrix A (the first factor of product AB on the matrix on R.H.S.)

Thus on the application of a sequence of row operations on the matrix equation $X = AB$ (these row operations are applied on X and on the first matrix A of product AB simultaneously), the matrix equation is still valid (we assume this fact without proof).

Similarly a sequence of elementary column operations on the matrix equation $X = AB$can be applied simultaneously on X and on the second matrix B of product AB and the equation will be still valid.

In view of the above mentioned fact, it is clear that we can find the inverse of a matrix A, if it exists, by using either a sequence of elementary row operations or a sequence of elementary column operations but not both simultaneously.

Using row operation

Apply a series of row operation on $A = IA$ till we get $I = BA$

Now by definition of inverse of a matrix $B = A^{-1}$

Using Column operation

Apply a series of column operations on $A = AI$ till we get $I = AB$. By definition of inverse B is inverse of A.

Illustration 8

Question: **Obtain the inverse of the matrix using elementary operations** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: **Using row operation**

$$A = IA, \text{ i.e., } \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \leftrightarrow R_2)$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \to R_3 - 3R_1)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_1 \to R_1 - 2R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \quad (\text{applying } R_3 \to R_3 + 5R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \quad (\text{applying } R_3 \to \frac{1}{2}R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \quad (\text{applying } R_1 \to R_1 + R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \quad (\text{applying } R_2 \to R_2 - 2R_3)$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Using column operation

$$A = AI\text{, i.e., } \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_1 \leftrightarrow C_2)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 - 2C_1)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 1/2 \\ 1 & 0 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad \left(C_3 \to \frac{1}{2}C_3\right)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} -2 & 1 & 1/2 \\ 1 & 0 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad (C_1 \to C_1 - 2C_2)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} 1/2 & 1 & 1/2 \\ -4 & 0 & -1 \\ 5/2 & 0 & 1/2 \end{bmatrix} \qquad (C_1 \to C_1 + 5C_3)$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} \qquad (C_2 \to C_2 - 3C_3)$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

14 RANK OF A MATRIX

A number r is said to be the rank of a matrix A if it possesses the following two properties:

(i) There is at least one square submatrix of A of order r whose determinant is not equal to zero.

(ii) If the matrix A contains any square submatrix of order $r + 1$, then the determinant of every square submatrix of A of order $r + 1$, should be zero.

In short the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

15 HOMOGENEOUS LINEAR EQUATIONS

The equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0\ , \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0\ , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{array}\right\} \qquad \ldots(i)$$

represents system of m homogeneous equations in n unknowns $x_1, x_2, \ldots\ldots, x_n$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}_{m\times 1},$$

where A, X, O are $m\times n$, $n\times 1$, $m\times 1$ matrices respectively. Then obviously we can write the system of equations (i) in the form of a single matrix equation

$AX = O$ …(ii)

The matrix A is called the coefficient matrix of the system of equations.

(i) If $|A| = 0$ the system has infinitely many solutions.

(ii) If $|A| \neq 0$ the system has zero solution or trivial solutions.

These conclusions can also be written on the basis of rank method as follows:

Suppose we have m equations in n unknowns. Then the coefficient matrix A will be of the type $m\times n$. Let r be the rank of the matrix A. Obviously r cannot be greater than n (the number of columns of the matrix A). Therefore we have either $r = n$ or $r<n$.

Case I: If $r = n$, the equation $AX = O$ will have $n-n$ i.e., no linearly independent solutions. In this case the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.

Case II: If $r<n$, we shall have $n-r$ linearly independent solutions. Any linear combination of these $n-r$ solutions will also be a solution of $AX = O$. Thus in this case the equation $AX = O$ will have an infinite number of solutions.

Case III: Suppose $m<n$ i.e., the number of solutions is less than the number of unknowns. Since $r\leq m$, therefore r is definitely less than n. Hence in this case the given system of equations must posses a non-zero solution. The number of solutions of the equation $AX = O$ will be infinite.

Illustration 9

Question: **Does the following system of equations possess a common non-zero solution?**

$$\mathbf{x + 2y + 3z = 0}$$
$$\mathbf{3x + 4y + 4z = 0}$$
$$\mathbf{7x + 10y + 12z = 0}$$

Solution: Determinant of coefficient matrix is $|A| = -2$ which non-zero

$\therefore\ x = y = z = 0$ is the only solution.

Alternate method : (using rank)

The given system of equations can be written in the form of the single matrix equation

$$AX = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O.$$

We shall start reducing the coefficient matrix A to triangular form by applying only E-row transformations on it. Applying $R_2\to R_2 - 3R_1$, $R_3\to R_3 - 7R_1$, the given system of equations is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

Here we find that the determinant of the matrix on the left hand side of this equation is not equal to zero. Therefore the rank of this matrix is 3. So there is no need of further applying E-row transformation on the coefficient matrix. The rank of the coefficient matrix A is 3, i.e., equal to the number of unknowns. Therefore the given system of equations does not possess any linearly independent solution. The zero solution, i.e. $x = y = z = 0$ is the only solution of the given system of equations.

The equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots.. + a_{1n}x_n = b_1 \;, \\ a_{21}x_1 + a_{22}x_2 + \ldots.. + a_{2n}x_n = b_2 \;, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(i)$$

be a system of m non-homogeneous equations in n unknowns $x_1, x_2, \ldots\ldots, x_n$. If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}_{m\times 1},$$

where A, X, B are $m\times n$, $n\times 1$, $m\times 1$ matrices respectively the above equations can be written in the form of a single matrix equation $AX = B$.

Any set of values of $x_1, x_2, \ldots.., x_n$ which simultaneously satisfy all these equations is called a solution of the system (i). When the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

If $B \neq 0$ the system (i) is said to be non-homogenous.

(i) If $|A| \neq 0 \quad \Rightarrow \quad X = A^{-1}B$, where $A^{-1} = \dfrac{Adj\ A}{|A|}$

the given system has unique solution.

(ii) If $|A| = 0$

$\because \quad AX = B$

$\Rightarrow \quad (adj\ A)AX = (adj\ A)B \quad \Rightarrow \quad |A|\,X = (adj\ A)B$

$\Rightarrow (adj\ A)B = 0 \qquad [\because |A| = 0]$

which is true for infinite values of X.

$\therefore$ for infinitely many solutions to the system $(adj\ A)B = 0$

Clearly for no solution $(adj\ A)B \neq 0$

These conclusions can also be written on the basis of rank method as follows:

The matrix $[A\ B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$

is called the augmented matrix of the given system of equations.

Suppose the coefficient matrix A is of the type $m\times n$, i.e., we have m equations in n unknowns. Write the augmented matrix $[AB]$ and reduce it to a Echelon form by applying only E-row transformations and comparing the ranks of the augmented matrix $[AB]$ and the coefficient matrix A. Then the following different cases arise:

Case I: Rank A < Rank $[AB]$

In this case the equations $AX = B$ are inconsistent i.e., they have no solution.

Case II: Rank A = Rank $[A\ \ B] = r$ (say).

In this case the equations $AX = B$ are consistent i.e., they possess a solution. If $r<m$, then in the process of reducing the matrix $[AB]$ to Echelon form, $(m-r)$ equations will then the replaced by an equivalent system of r equations. From these r equations we shall be able to express the values of some r unknowns in terms of the remaining $n-r$ unknowns which can be given any arbitrary chosen values.

If $r = n$, then $n-r = 0$, so that no variable is to be assigned arbitrary values and therefore in this case there will be a unique solution.

If $r<n$, then $n-r$ variables can be assigned arbitrary values. So in this case there will be an infinite number of solutions Only $n-r+1$ solutions will be linearly independent and the rest of the solutions will be linear combinations of them.

If $m<r$, then $r \le m < n$. Thus in this case $n-r > 0$. Therefore when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solution provided they are consistent.

For non singular matrix A

Equation $AX = B \quad \Rightarrow X = A^{-1}B$

By comparing entries on both the sides we have unique solution for given system of equations.

Illustration 10

Question: **Show that the equations $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$ are not consistent.**

Solution:

$$\Delta = |A| = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} -11 & 6 & 0 \\ -3 & 20 & -6 \\ -1 & 6 & -18 \end{vmatrix} \ne 0 ; \Delta_2 = \begin{vmatrix} 2 & -11 & 6 \\ 6 & -3 & 20 \\ 0 & -1 & 6 \end{vmatrix} \ne 0 ; \ \Delta_3 = \begin{vmatrix} 2 & 6 & -11 \\ 6 & 20 & -3 \\ 0 & 6 & -1 \end{vmatrix} \ne 0$$

$\Rightarrow$ the system is inconsistent

Alternate Method :

The given system of equations is equivalent to the single matrix equation

$$AX = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} = B$$

We shall reduce the coefficient matrix A to triangular form by E-row operations on it and apply the same operations on the right hand side i.e., on the matrix B.

Performing $R_2 \to R_2 - 3R_1$, we have, $\begin{bmatrix} 2 & 6 & 3 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$

Performing $R_3 \to R_3 - 3R_2$, we have, $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$

The last equation of this system is $0x + 0y + 0z = -91$. This shows that the given system is not consistent.

PROFICIENCY TEST–II

The following questions deal with the basic concepts of this section. Answer the following briefly. Do not consult the Study Material while attempting the questions.

1. Use matrix multiplication to divide Rs. 30,000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts Rs. 3060.

2. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, show that $AB = BA = O_{3\times 3}$.

3. Express the matrix $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

4. Let A and B be symmetric matrices of the same order. Then, show that
(ii) $AB - BA$ is a skew symmetric matrix
(iii) $AB + BA$ is a symmetric matrix

5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, show that $(AB)^T = B^T A^T$.

ANSWERS TO PROFICIENCY TEST–II

1. first part $\rightarrow$ 12,000
second part $\rightarrow$ 18,000

3. $$A=\begin{bmatrix}3 & 3 & 5/2\\ 3 & 5 & 7/2\\ 5/2 & 7/2 & 5\end{bmatrix}+\begin{bmatrix}0 & -1 & 1/2\\ 1 & 0 & -1/2\\ -1/2 & 1/2 & 0\end{bmatrix}$$

SOLVED OBJECTIVE EXAMPLES

Example 1:

If A is the diagonal matrix diag $(d_1, d_2, d_3, \ldots\ldots\ldots, d_n)$, then A^n, $n \in N$, is

(a) diag $(nd_1, nd_2, nd_3, \ldots\ldots\ldots, nd_n)$ **(b) diag $(d_1^n, d_2^n, d_3^n, \ldots\ldots\ldots, d_n^n)$**

(c) diag $(d_1^{n-1}, d_2^{n-1}, d_3^{n-1}, \ldots\ldots\ldots, d_n^{n-1})$ **(d) none of these**

Solution:

Given $A = \text{diag}\,(d_1, d_2, d_3, \ldots\ldots\ldots, d_n)$

Now $A^2 = AA$

$$= \begin{bmatrix} d_1 & 0 & 0 & .... & 0 \\ 0 & d_2 & 0 & .... & 0 \\ 0 & 0 & d_3 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & .... & 0 \\ 0 & d_2 & 0 & .... & 0 \\ 0 & 0 & d_3 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & 0 & .... & 0 \\ 0 & d_2^2 & 0 & .... & 0 \\ 0 & 0 & d_3^2 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n^2 \end{bmatrix}$$

and $A^3 = A^2A$

$$= \begin{bmatrix} d_1^2 & 0 & 0 & .... & 0 \\ 0 & d_2^2 & 0 & .... & 0 \\ 0 & 0 & d_3^2 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n^2 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & .... & 0 \\ 0 & d_2 & 0 & .... & 0 \\ 0 & 0 & d_3 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n \end{bmatrix} = \begin{bmatrix} d_1^3 & 0 & 0 & .... & 0 \\ 0 & d_2^3 & 0 & .... & 0 \\ 0 & 0 & d_3^3 & .... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .... & d_n^3 \end{bmatrix}$$

continuing this way, we get the result $A^n = \text{diag}\,(d_1^n, d_2^n, d_3^n, \ldots\ldots, d_n^n)$.

Hence (b) is the correct answer.

Example 2:

If A and B are any 2×2 matrices, then det $(A + B) = 0$ implies

(a) det A + det $B = 0$ **(b) det $A = 0$ or der $B = 0$**

(c) det $A = 0$ and det $B = 0$ **(d) none of these**

Solution:

det $(A + B)$ cannot be expressed in terms of det A and det B. Hence the given equation gives no inference.

Hence (d) is the correct answer.

Example 3:

If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ **and** $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$**, then**

(a) $a = 2, c = 1/2$ **(b) $a = 1, c = -1$**

(c) $a = -1, c = 1$ **(d) $a = 1/2, c = 1/2$**

Solution:

We must have $AA^{-1} = I$

$\therefore$ (3, 1)th entry of $AA^{-1} = 0 =$ (1, 3)th entry of AA^{-1}

$\Rightarrow \quad 3 \times \frac{1}{2} + a \times (-4) + 1 \times \frac{5}{2} = 0 = 0 \times \frac{1}{2} + 1 \times c + 2 \times \frac{1}{2}$

$\Rightarrow \quad -4a + 4 = 0$ and $c + 1 = 0 \quad \Rightarrow \quad a = 1$ and $c = -1$.

Hence (b) is the correct answer.

Example 4:

If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then α is

(a) 0 (b) ±5 (c) ±3 (d) ±2

Solution:

$A^3 = \begin{bmatrix} \alpha^3 + 12\alpha & 6\alpha^2 + 8 \\ 6\alpha^2 + 8 & \alpha^3 + 12\alpha \end{bmatrix}$; $|A^3| = 125$

$\Rightarrow (\alpha^2 - 4)^3 = 5^3$ $\Rightarrow \alpha = \pm 3$

Hence (c) is the correct answer.

Example 5:

Let $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2005} P$ is

(a) $\frac{1}{4}\begin{bmatrix} 1 & 2005 \\ -2005 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2005 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\frac{1}{4}\begin{bmatrix} -1 & -2005 \\ 2005 & 1 \end{bmatrix}$

Solution:

As $PP^T = I = P^T P$

$\Rightarrow P^T = P^{-1} \Rightarrow Q^n = PA^nP^{-1} \Rightarrow Q^{2005} = PA^{2005}P^{-1} \Rightarrow P^T Q^{2005} P = A^{2005}$

$= \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

Hence (b) is the correct answer.

SOLVED SUBJECTIVE EXAMPLES

Example 1:

If $\omega = e^{2i\pi/3}$, find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Solution:

Clearly $\omega^3 = e^{2\pi i} = 1$

Also $\omega + \omega^2 = e^{2\pi i/3} + e^{4\pi i/3} = (\cos 2\pi/3 + \cos 4\pi/3) + i\,(\sin 2\pi/3 + \sin 4\pi/3)$

$= 2\cos\pi \cos\pi/3 + i2 \sin\pi \cos\pi/3 = -1$ or $1 + \omega + \omega^2 = 0$

$|\,A\,| = (\omega^2-\omega^4) - (\omega-\omega^2) + (\omega^2-\omega) = 3\omega\,(\omega - 1) \neq 0.$

The cofactors of elements of various rows of | A | are

$(\omega^2-\omega^4)$, $-(\omega-\omega^2)$, $(\omega^2-\omega)$

$-(\omega-\omega^2)$, $(\omega - 1)$, $-(\omega^2 - 1)$

$(\omega^2-\omega)$, $-(\omega^2 - 1)$, $(\omega - 1)$

Put $\omega^4-\omega^3$, , $\omega = \omega - (\omega^2 - 1) = -(\omega + 1)(\omega - 1) = \omega^2(\omega - 1)$

$\therefore$ The matrix formed by cofactors of | A | is

$$C = \begin{bmatrix} \omega(\omega-1), & \omega(\omega-1) & \omega(\omega-1) \\ \omega(\omega-1) & \omega^3(\omega-1) & \omega^2(\omega-1) \\ \omega(\omega-1) & \omega^2(\omega-1) & \omega^3(\omega-1) \end{bmatrix}$$

$\therefore$ Adj. A = Transpose of matrix C of cofactors

$$= \omega(\omega-1)\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|\,A\,|}\text{ Adj. } A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

You can easily verify by using $1 + \omega + \omega^2 = 0$ that $AA^{-1} = A^{-1}A = I_3$.

Example 2:

Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$

Solution:

Let $B = \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$

We know that

$$B^{-1} = \frac{Adj.\,B}{|\,B\,|} = \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$$

$\therefore$ R.H.S.

$$= \cos^2\theta/2\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$$

$$= \cos^2\theta/2\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & -\tan^2\theta/2+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta/2-\sin^2\theta/2 & -2\sin\theta/2\cos\theta/2 \\ 2\sin\theta/2\cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \text{L.H.S.}$$

Example 3:

Show that the equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$, have no solution unless $a + b + c = 0$, in which case, they have infinitely many solutions. Find these solutions when $a = 1$, $b = 1$, $c = -2$.

Solution:

Writing the equations as $x + y - 2z = c$, $x - 2y + z = b$, $-2x + y + z = a$, and performing successively the operations $R_{21}(-1)$, $R_{31}(2)$, $R_{32}(1)$, we get

$$x + y - 2z = c,\quad -3y + 3z = b - c,\quad 0 = a + b + c,$$

Hence in order that the system be consistent, $a + b + c$ must be equal to zero.
In that case, from the first two equations we get

$$y = \frac{1}{3}c - \frac{1}{3}b + z\ ,\ x = c - y + 2z = \frac{2}{3}c + \frac{1}{3}b - z\ .$$

Substituting $b = 1$, $c = -2$, we get $x = y = z - 1$.
The number of solutions is infinite.

Example 4:

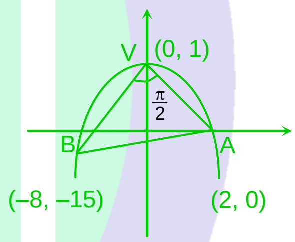
Let a, b, c are real and distinct numbers and $f(x)$ is a quadratic function such that

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

A is the point where $y = f(x)$ cuts the x-axis and B is the a point such that AB subtends a right angle at V (local maxima of $f(x)$). Find the area enclosed by the curve $y = f(x)$ and the chord AB.

Solution:

$$\left.\begin{aligned} 4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a \\ 4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b \\ 4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c \end{aligned}\right\} \Rightarrow f(-1) = \frac{3}{4} = f(1),\ f(2) = 0$$



Let $f(x) = Ax^2 + Bx + C$, using the value of $f(-1), f(1), f(2)$

$$\Rightarrow \quad f(x) = -\frac{1}{4}x^2 + 1$$

Let $B(2t, 1 - t^2)$ is any point on $f(x)$.

Given $m_{BV} \times m_{VA} = -1$

$$\Rightarrow \quad \frac{1 - t^2 - 1}{2t} \times \frac{1 - 0}{0 - 2} = -1$$

$\Rightarrow t = -4$,
so $B(-8, -15)$
Equation of line AB

$$\Rightarrow \quad 2y = 3x - 6$$

$$\Rightarrow \quad y = \frac{3}{2}x - 3.$$

Required area = $\int_{-8}^{2}\left(-\frac{1}{4}x^2 + 1\right)dx - \int_{-8}^{2}\left(\frac{3}{2}x - 3\right) = \frac{125}{3}$ square units, point A can also taken as $(-2, 0)$, then B will be $(8, -15)$ and area will be same.

Example 5:

If D = diag . [d_1, d_2, …… , d_n], d_1, d_2, ….. , $d_n \neq 0$, the prove that D^{-1} = diag. $[d_1^{-1}, d_2^{-1}, \ldots d_n^{-1}]$.

Solution:

We have $D = \begin{bmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \ldots & d_n \end{bmatrix}$.

Since $|D| = d_1, d_2, \ldots. d_n \neq 0$, D is non-singular.

Now adj. $D = \begin{bmatrix} d_2 d_3 \ldots d_n & 0 & 0 & \ldots. & 0 \\ 0 & d_1 d_3 \ldots d_n & 0 & \ldots. & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \ldots & d_1 d_2 \ldots. d_{n-1} \end{bmatrix}$

Therefore $D^{-1} \dfrac{1}{|D|}$ adj. D = diag. $\left[\dfrac{1}{d_1}.\dfrac{1}{d_2}, \ldots\ldots, \dfrac{1}{d_n}\right]$

MIND MAP

MULTIPLICATION OF MATRICES

- ProductAB is possible if matrix A has order $m \times n$ and B has the order $n \times l$.
- AB = $[C_{ij}]_{m\times l}$ where $C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.
- AB $\neq$ BA (in general)
- A(BC) = (AB)C
- A(B + C) = AB + AC

KINDS OF MATRICES DEPENDING UPON THEIR ENTRIES

- Null matrix : If $a_{ij} = 0$, $\forall$ i and j.
- Square matrix : If m = n.
- Unit matrix : Square matrix having $a_{ij} = 0$, $\forall$ $i \neq j$ and $a_{ij} = 1$ for i = j.
- Row matrix : If m = 1
- Column matrix : If n = 1
- Diagonal matrix : Square matrix $a_{ij} = 0$, $\forall$ $i \neq j$ and $a_{ij} \neq 0$ for i = j
- Upper triangular matrix : If $a_{ij} = 0$ for i > j.
- Lower triangular matrix : If $a_{ij} = 0$ for i < j.

MINOR

- Minor M_{ij} of any element a_{ij} of the matrix A is equal to the determinant of matrix leaving i^{th} row and j^{th} column.

COFACTOR

- Cofactor C_{ij} of any element a_{ij} of the matrix $= (-1)^{i+j} M_{ij}$

ADJOINT OF MATRIX

- For square matrix A adj A = $[C_{ij}]^T$

INVERSE OR RECIPROCAL OF A NON SINGULAR MATRIX A

- $(A^{-1}) = \dfrac{adj\ A}{|A|}$

OPERATIONS RELATED TO MATRICES

- Transpose $(A^T) = [a_{ji}]_{n\times m}$
- Conjugate $(\bar{A}) = [\bar{a}_{ij}]_{m\times n}$
- Transpose of conjugate $(A^{\theta}) = [\bar{a}_{ji}]_{n\times m}$
- Determinant of matrix | A | (for square matrix A)

  $= \sum_{k=1}^{n} a_{ik} C_{ik}$ where $\{1 \leq i \leq m\}$
- Trace of matrix $T_r(A) = \sum_{i=1}^{n} a_{ii}$

MATRIX

$$A = [a_{ij}]_{m\times n},\ 1 \leq i \leq m\ ,\ 1 \leq j \leq n$$

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

- Let the equations be

  $a_{11}x + a_{12}y + a_{13}z = b_1$

  $a_{21}x + a_{22}y + a_{23}z = b_2$

  $a_{31}x + a_{32}y + a_{33}z = b_3$
- These equations can be written in matrix form as

  AX = B

  $\Rightarrow$ $X = A^{-1}B$

  where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$, $A = [a_{ij}]_{3\times 3}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1}$

KINDS OF MATRICES DEPENDING UPON OPERATIONS

- Singular matrix : Square matrix having |A| = 0
- Non singular matrix : Square matrix having $|A| \neq 0$.
- Symmetric : If $a_{ij} = a_{ji}$
- Skew symmetric : if $a_{ij} = -a_{ji}$
- Hermitian : If $a_{ij} = \bar{a}_{ji}$
- Skew hermitian : If $a_{ij} = -\bar{a}_{ji}$
- Orthogonal : Square matrix for which $A^T A = I$
- Unitary : Square matrix for which $A^{\theta} A = I$
- Idempotent : If $A^2 = A$
- Periodic : If $A^{k+1} = A$, k is positive integer known as period of A.
- Nilpotent : If $A^{\lambda} = 0$, λ is positive integer known as index of A.

EXERCISE – I

CBSE PROBLEMS

1. If $A=\begin{bmatrix}1 & 3\\ 2 & 1\\ 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 1 & 2\\ 1 & 1\end{bmatrix}$, then find the matrix C such that $A+B+C$ is a zero matrix.

2. Find the value of x and y from the equation $2\begin{bmatrix}x & 5\\ 7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 14\\ 15 & 14\end{bmatrix}$.

3. Find a matrix X, such that $2A+B+X=0$, where $A=\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$.

4. If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find k, such that $A^2=kA-2I$.

5. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, prove that $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$, $n\in N$.

6. (i) If $A=\begin{bmatrix}3 & -4\\ 7 & 8\end{bmatrix}$, show that $A-A'$ is a skew-symmetric matrix.

(ii) If A and B are symmetric matrices of same order, then show that $AB-BA$ is skew-symmetric matrix.

7. If $A=\begin{bmatrix}3 & -5\\ -4 & 2\end{bmatrix}$, verify that $A^2-5A-14I=O$ and hence find A^{-1}.

8. Solve the equation using matrix-method:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,\ \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2.$$

9. **(i)** If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$, find A^{-1}, hence solve the equations

$x+y+2z=0,\ x+2y-z=9,\ \ x-3y+3z=-14$.

(ii) Using elementary transformation find the inverse of $\begin{bmatrix}2 & -3 & 3\\ 2 & 2 & 3\\ 3 & -2 & 2\end{bmatrix}$.

10. If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$, find AB and use this result to solve the equations $2x-y+z=-1,\ -x+2y-z=4,\ x-y+2z=-3$.

11. Solve the equations $x+2y+z=7,\ x+3z=11,\ 2x-3y=1$.

12. If $A=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$, show that $A^2-4A-I_2=O$. Hence find A^{-1}.

13. If $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$, show that $A'A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$.

14. Compute the adjoint of the matrix $A=\begin{bmatrix}3 & 4\\ 5 & 7\end{bmatrix}$ and verify that $(adj\ A)A=|A|\,I$.

15. Find the value of x, such that $\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=O$.

EXERCISE – II

IIT-JEE-SINGLE CHOICE CORRECT

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 2 & 1 \end{bmatrix}$ and $A + B - C = 0$, then $C =$

(a) $\begin{bmatrix} 5 & 5 \\ 6 & 8 \\ 7 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -4 & -4 \\ 3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 5 \\ 8 & 6 \\ 7 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 5 \\ 8 & 7 \\ 7 & 5 \end{bmatrix}$

2. If $\begin{bmatrix} 5 & k+2 \\ k+1 & -2 \end{bmatrix} = \begin{bmatrix} k+3 & 4 \\ 3 & -k \end{bmatrix}$ then $k =$

(a) 0 (b) 2
(c) –2 (d) 1

3. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, then $(A - 2I)(A - 3I) =$

(a) A (b) I
(c) O (d) 5I

4. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, then Adj $A =$

(a) A (b) A^T
(c) 3A (d) $3A^T$

5. If A is $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & \lambda \end{bmatrix}$ is a singular matrix then $\lambda =$

(a) 3 (b) 4
(c) 2 (d) 5

6. The inverse of the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$ is

(a) $-A$ (b) A

(c) $\begin{bmatrix} -\cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix}$ (d) none of these

7. If A is 3 × 3 matrix and B is a matrix such that $A'B$ and BA' are both defined. Then B is of the type

(a) 3 × 4 (b) 3 × 3
(c) 4 × 4 (d) 4 × 3

8. For any 2 × 2 matrix A, if $A(\text{Adj. } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ equals

(a) 0 (b) 10
(c) 20 (d) 100

9. If $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $E(\alpha) . E(\beta)$ is equal to

(a) $E(0)$ (b) $E(\alpha\beta)$
(c) $E(\alpha + \beta)$ (d) $E(\alpha - \beta)$

10. If $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $x^n, n \in N$ is equal to

(a) $2^{n-1}x$ (b) n^2x
(c) nx (d) $2^{n+1}x$

11. Choose the incorrect

(a) $A^2 - B^2 = (A+B)(A-B)$ (b) $(AB)^n = A^nB^n$, where A, B commute
(c) $(A^T)^T = A$ (d) $(A-I)(A+I) = 0 \Leftrightarrow A^2 - I = 0$

12. The inverse of a diagonal matrix is
(a) a symmetric matrix (b) a skew symmetric matrix
(c) a diagonal matrix (d) none of these

13. If A is an orthogonal matrix, then $A^{-1} =$
(a) A (b) A^2
(c) A' (d) none of these

14. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, then $pI + qA + rA^2$

(a) A (b) $2A$
(c) $3A$ (d) A^3

15. The matrix 'X' in the equation $AX = B$, such that $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is given by

(a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ -3 & 1 \end{bmatrix}$

16. For the equations $x + 2y + 3z = 1$, $2x + y + 3z = 2$, $5x + 5y + 9z = 4$
(a) there is only one solution (b) there exist infinitely many solutions
(c) there is no solution (d) none of these

17. If $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$, $G(y)=\begin{bmatrix}\cos y & 0 & \sin y\\ 0 & 1 & 0\\ -\sin y & 0 & \cos y\end{bmatrix}$, then $[F(x) . G(y)]^{-1}$ is equal to

(a) $F(x)\ G(-y)$
(b) $F(x^{-1})\ G(y^{-1})$
(c) $G(y^{-1})\ F(x^{-1})$
(d) $G(-y)\ F(-x)$

18. If $A=\begin{bmatrix}1 & 3\\ 3 & 4\end{bmatrix}$ and $A^2-kA-5I_2=0$, then the value of k is

(a) 3
(b) 5
(c) 7
(d) none of these

19. If (1, 2, 3) B = (3, 4), then order of B is

(a) 3×1
(b) 1×3
(c) 2×3
(d) 3×2

20. If $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$, then

(a) $A^2 = B^2 = I$
(b) $A^2 = B^2 = -I$
(c) $A^2 = I$, $B^2 = -I$
(d) $A^2 = -I$, $B^2 = -I$

21. If $A=\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$ such that $A^2=B$, then α is

(a) 1
(b) –1
(c) 4
(d) none of these

22. If $f(x)=x^2+4x-5$ and $A=\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}$, then $f(A)$ is equal to

(a) $\begin{bmatrix}0 & -4\\ 8 & 8\end{bmatrix}$
(b) $\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}$
(c) $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$
(d) $\begin{bmatrix}8 & 4\\ 8 & 0\end{bmatrix}$

23. If A is an invertible symmetric matrix, then A^{-1} is

(a) symmetric
(b) skew symmetric
(c) a diagonal matrix
(d) none of these

24. If $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, I is the unit matrix of order 2 and a, b are arbitrary constants, then $(aI+bA)^2$ is equal to

(a) a^2I+abA
(b) $a^2I+2abA$
(c) a^2I+b^2A
(d) none of these

25. Matrix A is such that $A^2=2A-I$, where I is the identity matrix. Then for $n\ge 2$, A^n is equal to

(a) $nA-(n-1)I$
(b) $nA-I$
(c) $2^{n-1}A-(n-1)I$
(d) $2^{n-1}A-I$

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. If $[1\ \ x\ \ 1]\begin{bmatrix}1 & 3 & 2\\0 & 5 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix}=O$, then x equals to

(a) $-3\pm\sqrt{3}$ (b) $\dfrac{-9\pm\sqrt{53}}{2}$

(c) 1 (d) none of these

2. If $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that $ad-bc\neq 0$, then A^{-1} is equal to

(a) $\dfrac{1}{ad-bc}\begin{bmatrix}d & b\\-c & a\end{bmatrix}$ (b) $\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

(c) $\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$ (d) none of these

3. If A and B are symmetric matrices of the same order, then

(a) AB is a symmetric matrix

(b) $A-B$ is a skew-symmetric matrix

(c) $AB+BA$ is a symmetric matrix

(d) $AB-BA$ is a symmetric matrix

4. If $A=[x\ \ y\ \ z]$, $B=\begin{bmatrix}a & h & g\\h & b & f\\g & f & c\end{bmatrix}$ and $C=[x\ \ y\ \ z]^t$, then ABC is

(a) not defined (b) a 3×3 matrix

(c) a 1×1 matrix (d) none of these

5. If $A=\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 4\end{bmatrix}$, then $adj(adj\ A)$ is equal to

(a) A (b) $-A$

(c) A^2 (d) none of these

6. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is

(a) 18 (b) 512

(c) 81 (d) none of these

7. If $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, then A^n is equal to

(a) $\begin{bmatrix}1 & 2n\\0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}2 & n\\0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}1 & 2n\\0 & -1\end{bmatrix}$ (d) $\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$

8. The value of 'a' for which the system of equations

$$a^3x+(a+1)^3y+(a+2)^3z=0,\ ax+(a+1)y+(a+2)z=0 \text{ and } x+y+z=0$$

has a non-zero solution is

(a) 1 (b) 0 (c) –1 (d) none of these

9. If $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, then for any integer '$n$' the value of $\lim\limits_{n\to\infty}\dfrac{1}{n^2}A^{-n}$ is

(a) $\begin{bmatrix}0&0\\-1&0\end{bmatrix}$ (b) $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ (c) $\begin{bmatrix}0&0\\0&-1\end{bmatrix}$ (d) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$

10. If A and B are square matrices of size $n \times n$ such that $A^2-B^2=(A-B)(A+B)$, then which of the following will be always true?

(a) $AB = BA$
(b) either of A or B is a zero matrix
(c) either of A or B is an identity matrix
(d) $A = B$

11. If $A^3=0$ but $A^n \neq 0$ for n = 1, 2 then $(I-A)^{-1}$ is equal to

(a) $1+A+A^2$ (b) $1-A+A^2$ (c) $1-A-A^2$ (d) $1+A-A^2$

12. If $A=\begin{bmatrix}0&5\\0&0\end{bmatrix}$ and $f(x)=I+x+x^2+......+x^{16}$, then $f(A)=$

(a) 0 (b) $\begin{bmatrix}1&5\\0&1\end{bmatrix}$ (c) $\begin{bmatrix}1&5\\0&0\end{bmatrix}$ (d) $\begin{bmatrix}1&5\\1&1\end{bmatrix}$

13. If $A=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, then sum of the series $S_k=I+2A+3A^2+4A^3+......k$ terms is

(a) $\begin{bmatrix}k&1\\1&k\end{bmatrix}$ (b) $\begin{bmatrix}1&0\\0&\sum\limits_{r=1}^{k-1}r\end{bmatrix}$ (c) $\begin{bmatrix}-1&0\\0&\sum\limits_{r=1}^{k}r\end{bmatrix}$ (d) $\begin{bmatrix}1&0\\0&\sum\limits_{r=1}^{k}r\end{bmatrix}$

14. If $AB = 0$, where $A=\begin{bmatrix}\cos^2\theta&\cos\theta\sin\theta\\\cos\theta\sin\theta&\sin^2\theta\end{bmatrix}$ and $B=\begin{bmatrix}\cos^2\phi&\cos\phi\sin\phi\\\cos\phi\sin\phi&\sin^2\phi\end{bmatrix}$, then $|\theta-\phi|$ is equal to

(a) 0 (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) π

15. If $A=\begin{bmatrix}1&0&0\\0&1&1\\0&-2&4\end{bmatrix}$ and $A^{-1}=\dfrac{1}{6}(A^2+cA+dI)$, then ($c$, d) is

(a) (–6, –11)
(b) (–6, 11)
(c) (6, –11)
(d) (6, 11)

16. The values of λ and μ for which the system of equations

$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$

have no solution are

(a) $\lambda = 3, \mu \neq 0$
(b) $\lambda = 3, \mu = 10$
(c) $\lambda \neq 3, \mu = 10$
(d) $\lambda = 3, \mu \neq 10$

17. If $A = (a_{ij})$ is a 4 × 4 matrix and C_{ij} is the co-factor of the element a_{ij} in | A |, then the expression $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$ is equal to

(a) 0 (b) –1
(c) 1 (d) | A |

18. If $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta).E(\phi)$ is a

(a) null matrix (b) unit matrix
(c) diagonal matrix (d) none of these

19. Matrix $A_\lambda = \begin{bmatrix} \lambda & \lambda-1 \\ \lambda-1 & \lambda \end{bmatrix}$, $\lambda \in N$. The value of $|A_1| + |A_2| + ... + |A_{300}|$ is

(a) $(299)^2$ (b) $(300)^2$
(c) $(301)^2$ (d) none of these

20. If the matrix $\begin{bmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{bmatrix}$ is a zero matrix, then *a, b, c, x, y, z* are connected by

(a) $a+b+c=0,\ x+y+z=0$ (b) $a+b+c=0,\ x=y=z$
(c) $a=b=c,\ x+y+z=0$ (d) none of these

21. Let $A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta & -\cos\alpha\cos\beta & 0 \end{pmatrix}$, then

(a) | A | is independent of α and β (b) A^{-1} depends only on α
(c) A^{-1} depends only on β (d) none of these

22. If the system of equations $x = a(y+z)$, $y = b(z+x)$, $z = c(x+y)$ $(a, b, c \neq -1)$ has a non-trivial solution, then the value of $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c}$ is

(a) 1 (b) 0
(c) –1 (d) 2

23. If the system of equations $x + 4ay + az = 0$; $x + 3by + bz = 0$ and $x + 2cy + cz = 0$ have a non-zero solution, then *a, b, c* are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

24. If $A + B = 2B^T$ and $3A + 2B = I$, where A and B are matrices of same order then $5A + 5B$ is equal to

(a) I (b) $2I$
(c) $-I$ (d) $-2I$

25. If A is any square matrix, then $adj\,(A') - (adj\,A)'$ is equal to

(a) 2 $|A|$ (b) $|2A|$
(c) a null matrix (d) none of these

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. The system of equations $ax + by = c$, $a'x + b'y = c'$

(a) has a unique solution if $ab' - a'b \neq 0$

(b) has no solution if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$

(c) has infinite solutions if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(d) none of these

2. If A and B are invertible matrices both of order n, then

(a) adj. $A = |A|A^{-1}$

(b) $(A + B)^{-1} = A^{-1} + B^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) $|A^{-1}| = |A|^{-1}$

3. If $A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, then

(a) $|A| = 2$

(b) adj. $A = \begin{bmatrix} 1/2 & 0 & -1 \\ -1 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{bmatrix}$

(c) | adj. A | = 4

(d) $|A^{-1}| = 1/2$

4. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then

(a) $A^3 = I$

(b) $A^{-1} = A^2$

(c) $A^n = A \ \forall\, n \neq 4$

(d) none of these

5. The correct statement/s are

(a) A is symmetric $\Rightarrow A^n$ is symmetric $\forall\, n \in N$

(b) A is symmetric $\Rightarrow A^n$ is skew symmetric $\forall\, n \in N$

(c) A is skew symmetric $\Rightarrow A^n$ is symmetric $\forall\, n \in$ even N

(d) A is skew symmetric $\Rightarrow A^n$ is skew symmetric $\forall\, n \in$ odd N

6. If A is a non-singular matrix, then

(a) A^{-1} is symmetric if A is symmetric

(b) A^{-1} is skew symmetric if A is symmetric

(c) $|(A^{-1})'| = |A'|^{-1}$ if A is symmetric

(d) none of these

7. If A and B commutative, so also do

(a) A^{-1} and B^{-1}

(b) A' and B'

(c) $\overline{A}'$ and $\overline{B}'$

(d) none of these

8. Inverse of

(a) a non singular diagonal matrix diag $(k_1, k_2, ..., k_n)$ is the diagonal matrix diag $\left(\frac{1}{k_1}, \frac{1}{k_2}, ..., \frac{1}{k_n}\right)$.

(b) a non singular diagonal matrix diag $(k_1, k_2, ..., k_n)$ is the inverse of diag $\left(\frac{1}{k_1}, \frac{1}{k_2}, ..., \frac{1}{k_n}\right)$.

(c) product of two matrices is same as product of inverses of these matrices taken in reverse order.

(d) none of these.

9. If A is invertible matrix, then $\det(A^{-1})$ is not equal to

(a) $\det(A)$

(b) $\dfrac{1}{\det(A')}$

(c) $\dfrac{1}{\det(A)}$

(d) $\det(A')$

10. Let A be a skew-symmetric matrix of odd order, then $\det(A)$ cannot be

(a) 0

(b) ± 1

(c) ± 2

(d) ± 3

EXERCISE – V

MATCH THE FOLLOWING

Note: Each statement in column – I has one or more than one match in column - II

1.

Column I	Column II
I. If *A, B* and *C* are square matrices of order 3 such that $C = 3A$, $\lvert A \rvert = -2$, $\lvert B \rvert = 3$ and $\lvert CB \rvert = -162x$, then $(\lvert AB \rvert, x+10)$ is equal to	**A.** (–6, 11)
II. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $A^{-1} = \frac{1}{6}\left[A^2 + cA + dI\right]$, where $c, d \in R$, than pair of values (*c*, *d*) may be	**B.** (–1, 2)
III. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & -1 \\ -15 & 5\alpha - \alpha^2 & -5 \\ 5 & -2 & 2 \end{bmatrix}$. If *B* is inverse of matrix *A*, then $(\alpha, 3\lvert A \rvert)$ is	**C.** (2, 3)
IV. $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$. If $A^{40} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and $y = \alpha^3 + \beta^3 + \gamma^3 + \delta^3$, then $(\lvert A \rvert, y)$ is equal to	**D.** (0, 3)

Note: Each statement in column – I has one or more than one match in column - II

2.

Column II	Column II
I. If $C_k = {}^nC_k$ for $0 \le k \le n$ and $A_k = \begin{bmatrix} C_{k-1}^2 & 0 \\ 0 & C_k^2 \end{bmatrix}$ for $k \ge 1$ and $A_1 + A_2 + \ldots A_n = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$, then $k_1 - k_2$ is equal to	**A.** 1
II. If *A* is a square matrix such that $A^2 = A$, than det $\{(I+A)^3 - 7A\}$ is equal to	**B.** 0
III. If $\begin{bmatrix} 1 & -\tan\alpha \\ \tan\alpha & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\alpha \\ -\tan\alpha & 1 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$, then maximum value of *a* + *b* is	**C.** 5
IV. In the above question, if $m \le a^2 - b^2 \le M$, then $m^2 + M^2$ is equal to	**D.** 2

Note: Each statement in column – I has one or more than one match in column - II

3.

Column I	**Column II**
I. If A is a square matrix of order $n \times n$ and k is a scalar, then adj (kA) is equal to	**A.** $k^n \mid A \mid$
II. If $A = [a_{ij}]$ is scalar matrix of order $n \times n$ such that $a_{ii} = k$ for all i, then trace of A is equal to	**B.** $\mid A \mid^{n-2} A$
III. If A is a square matrix of order $n \times n$, then adj (adj A) is equal to	**C.** k^{n-1} adj$A$
IV. If $A = [a_{ij}]$ is a square matrix of order $n \times n$ and k is a scalar, then $\mid kA \mid$ =	**D.** nk

REASONING TYPE

Directions: Read the following questions and choose

(A) If both the statements are true and statement-2 is the correct explanation of **statement-1.**

(B) If both the statements are true but statement-2 is not the correct explanation of **statement-1.**

(C) If statement-1 is True and **statement-2** is False.

(D) If statement-1 is False and **statement-2** is True.

1. Statement-1: Let A be a non-singular square matrix of order n, then | adj A | is equal to $| A |^{n-1}$.

Statement-2: If A is a singular matrix, then A adj A is null matrix.

(a) A (b) B (c) C (d) D

2. Statement-1: A is a singular matrix, then adj A is singular.

Statement-2: $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then the value of | adj A | is equal to a^{27}.

(a) A (b) B (c) C (d) D

3. Statement-1: A is not a square matrix and $a_{ij} = a_{ji}$.

Statement-2: A cannot be symmetric.

(a) A (b) B (c) C (d) D

4. Statement-1: If A and B are two matrices so that $AB = B$ and $BA = A$, then $A^2 + B^2 = A + B$.

Statement-2: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then A^2 is equal to I_3.

(a) A (b) B (c) C (d) D

5. Statement-1: If $A = [a_{ij}]_{3\times 3}$ so that, $a_{ij} = \begin{cases} 0 \ \forall\ i \neq j \\ a \ \forall\ i = j \end{cases}$, then $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$.

Statement-2: If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then | adj (adj A) | = $(12)^4$.

(a) A (b) B (c) C (d) D

LINKED COMPREHENSION TYPE

I. If A be a square matrix of order n. Then
A (adj A) = | A | I_n = (adj A) A

II. If A and B are non-singular square matrices of the same order, then
adj AB = (adj B) (adj A)

III. If A be a non-singular square matrix of order n. Then
| adj A | = | A $|^{n-1}$.

IV. If A is a non-singular square matrix, then
adj (adj A) = | A $|^{n-2}A$

1. If A is a square matrix, then adj A^T–(adj A)^T is equal to

(a) 2 | A |
(b) 2 | A | I
(c) null matrix
(d) unit matrix

2. Let A be a non-singular square matrix. Then det (Adj Adj A) is equal to

(a) | A $|^n$
(b) | A $|^{(n-1)^2}$
(c) | A $|^{n-2}$
(d) none of these

3. If A = [a_{ij}] is a scalar matrix of order $n \times n$ such that $a_{ii} = k$ for all i, then | A | is equal to

(a) nk
(b) $n + k$
(c) n^k
(d) k^n

4. A is a non-singular square matrix of order n, then det (adj (det(A))) is equal to

(a) | A $|^{n-3}A$
(b) | A $|^{n-4}A$
(c) exist but cannot be evaluated
(d) does not exist.

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. Prove that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

2. If $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, prove that $(aI + bA)^n = a^nI + na^{n-1}bA$, where I is the two rowed unit matrix and n is a positive integer.

3. Let $f(x) = x^2 - 5x + 6$, find $f(A)$ i.e., $A^2 - 5A + 6I$, if $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$.

4. Find the inverse of the matrix $S=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ and show that SAS^{-1} is a diagonal matrix where $A=\frac{1}{2}\begin{bmatrix}b+c & c-a & b-a\\ c-b & c+a & a-b\\ b-c & a-c & a+b\end{bmatrix}$.

5. If A is a non-singular matrix, then show that adj (adj A) = $|A|^{n-2}A$.

6. If matrix $A=\begin{bmatrix}a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, then find the value of $a^3+b^3+c^3$.

7. If M is a 3 × 3 matrix where $|M| = 1$ and $MM^T = I$ where I is an identity matrix, prove that $|M-I|=0$.

8. If $A=\begin{bmatrix}a & 1 & 0\\ 1 & b & d\\ 1 & b & c\end{bmatrix}$, $B=\begin{bmatrix}a & 1 & 1\\ 0 & d & c\\ f & g & h\end{bmatrix}$, $U=\begin{bmatrix}f\\ g\\ h\end{bmatrix}$, $V=\begin{bmatrix}a^2\\ 0\\ 0\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $AX=U$ has infinitely many solutions. Prove that $BX=V$ has no unique solution. Also show that if $afd \neq 0$, then $BX=V$ has no solution.

9. For what values of the parameter λ will be following equations fail to have unique solution

$$3x-y+\lambda z = 1,\ 2x+y+z = 2,\ x+2y-\lambda z = -1?$$

Will the equations have any solutions for these values of λ?

10. Solve the equations

$$\lambda x + 2y - 2z - 1 = 0,\ 4x + 2\lambda y - z - 2 = 0,\ 6x + 6y + \lambda z - 3 = 0,$$

considering specially the case when $\lambda = 2$.

ANSWERS

EXERCISE – I

CBSE PROBLEMS

1. $\begin{bmatrix} -3 & -4 \\ -3 & -3 \\ -4 & 0 \end{bmatrix}$

2. $x = 2,\ y = 9$

3. $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$

4. $k = 1$

7. $A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & -5 \\ -4 & -3 \end{bmatrix}$

8. $x = 2, y = 3, z = 5$

9. **(i)** $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$; $x = 1, y = 3, z = -2$ **(ii)** $\begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix}$

10. $AB = 4I,\ x = 1,\ y = 2,\ z = -1$

11. $x = 2, y = 1, z = 3$

12. $A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

14. $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$

15. $x = -1$

EXERCISE – II

IIT-JEE-SINGLE CHOICE CORRECT

1. (a)	2. (b)	3. (c)	4. (d)	5. (a)
6. (b)	7. (b)	8. (b)	9. (c)	10. (a)
11. (a)	12. (c)	13. (c)	14. (d)	15. (b)
16. (a)	17. (d)	18. (b)	19. (d)	20. (d)
21. (d)	22. (d)	23. (a)	24. (b)	25. (a)

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. (b)	2. (c)	3. (c)	4. (c)	5. (b)
6. (b)	7. (a)	8. (c)	9. (b)	10. (a)
11. (a)	12. (b)	13. (d)	14. (b)	15. (b)
16. (a)	17. (d)	18. (a)	19. (b)	20. (d)
21. (a)	22. (a)	23. (c)	24. (b)	25. (c)

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. (a, b, c)	2. (a, b, c, d)	3. (b, d)	4. (a, b)	5. (a, c, d)
6. (a, c)	7. (a, b, c)	8. (a, c)	9. (a, d)	10. (b, c, d)

EXERCISE – V

MATCH THE FOLLOWING

1. I-[A], II-[A],[C], III-[C], [D] IV-[B]

2. I-[B], II-[A], III-[D], IV-[C]

3. I-[C], II-[D], III-[B], IV-[A]

REASONING TYPE

1. (b)	2. (c)	3. (d)	4. (b)	5. (c)

LINKED COMPREHENSION TYPE

1. (c)	2. (b)	3. (d)	4. (d)

EXERCISE – VI

SUBJECTIVE PROBLEMS

3. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

6. 4

9. $\lambda = -7/2$; No.

10. $\left.\begin{aligned} x &= 1/2 - c \\ y &= c \\ z &= 0 \end{aligned}\right\}$ c is arbitrary constant